ACCESSIBLE INDEPENDENCE RESULTS FOR PEANO ARITHMETIC

LAURIE KIRBY AND JEFF PARIS

Recently some interesting first-order statements independent of Peano Arithmetic (P) have been found. Here we present perhaps the first which is, in an informal sense, purely number-theoretic in character (as opposed to metamathematical or combinatorial). The methods used to prove it, however, are combinatorial. We also give another independence result (unashamedly combinatorial in character) proved by the same methods.

The first result is an improvement of a theorem of Goodstein [2]. Let m and n be natural numbers, $n > 1$. We define the *base n representation* of m as follows:

First write m as the sum of powers of n. (For example, if $m = 266$, $n = 2$, write $266 = 2^8 + 2^3 + 2^1$.) Now write each exponent as the sum of powers of n. (For example, $266 = 2^{2^3} + 2^{2+1} + 2^1$.) Repeat with exponents of exponents and so on until the representation stabilizes. For example, 266 stabilizes at the representation $2^{2^{2+1}} + 2^{2+1} + 2^1$.

We now define the number $G_n(m)$ as follows. If $m = 0$ set $G_n(m) = 0$. Otherwise set $G_n(m)$ to be the number produced by replacing every n in the base n representation of m by $n+1$ and then subtracting 1. (For example, $G_2(266) = 3^{3^{3+1}} + 3^{3+1} + 2$).

Now define the Goodstein sequence for m starting at 2 by

$$m_0 = m,\ m_1 = G_2(m_0),\ m_2 = G_3(m_1),\ m_3 = G_4(m_2),\ \dots .$$

So, for example,

$$266_0 = 266 = 2^{2^{2+1}} + 2^{2+1} + 2$$

$$266_1 = 3^{3^{3+1}} + 3^{3+1} + 2 \sim 10^{38}$$

$$266_2 = 4^{4^{4+1}} + 4^{4+1} + 1 \sim 10^{616}$$

$$266_3 = 5^{5^{5+1}} + 5^{5+1} \sim 10^{10{,}000} .$$

Similarly we can define the Goodstein sequence for m starting at n for any $n > 1$.

THEOREM 1. (i) (Goodstein [2]) $\forall m\, \exists k\, m_k = 0$. *More generally for any $m, n > 1$ the Goodstein sequence for m starting at n eventually hits zero.*

(ii) $\forall m\, \exists k\, m_k = 0$ (*formalized in the language of first order arithmetic*) *is not provable in P*.

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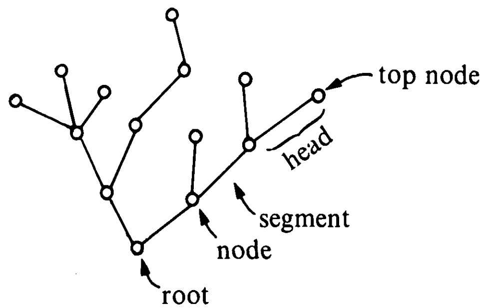
So, belying its early form, the sequence m_k eventually hits zero. However despite this fact being expressible in first order arithmetic we cannot give a proof of it in Peano Arithmetic P. As we shall see later the reason for this is the immense time it takes for the sequence m_k to reach zero. (For example, the sequence 4_k first reaches zero when $k = 3 \times 2^{402,653,211} - 3$, which is of the order of $10^{121,210,700}$.)

Before proving Theorem 1 we state our second result.

A *hydra* is a finite tree, which may be considered as a finite collection of straight line segments, each joining two nodes, such that every node is connected by a unique path of segments to a fixed node called the *root*. For example:

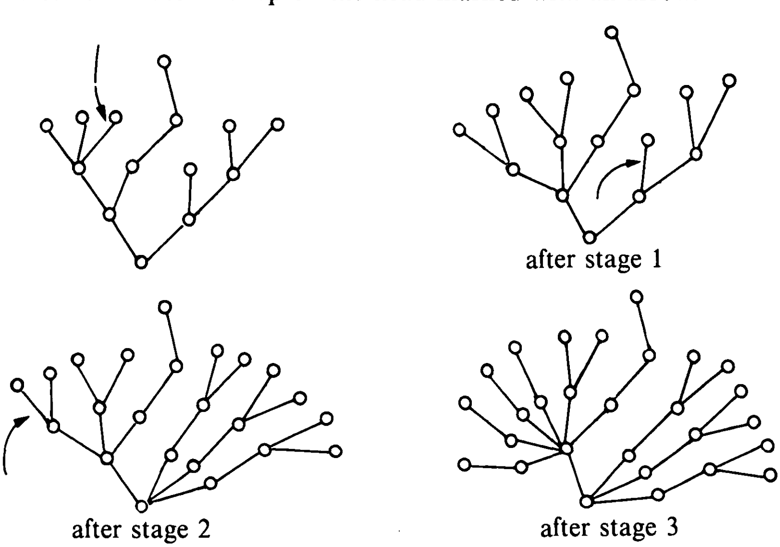


A *top node* of a hydra is one which is a node of only one segment, and is not the root. A *head* of the hydra is a top node together with its attached segment.

A battle between Hercules and a given hydra proceeds as follows: at stage n ($n \geqslant 1$), Hercules chops off one head from the hydra. The hydra then grows n "new heads" in the following manner:

From the node that used to be attached to the head which was just chopped off, traverse one segment towards the root until the next node is reached. From this node sprout n replicas of that part of the hydra (after decapitation) which is "above" the segment just traversed, i.e., those nodes and segments from which, in order to reach the root, this segment would have to be traversed. If the head just chopped off had the root as one of its nodes, no new head is grown.

Thus the battle might for instance commence like this, assuming that at each stage Hercules decides to chop off the head marked with an arrow:



Hercules *wins* if after some finite number of stages, nothing is left of the hydra but its root. A *strategy* is a function which determines for Hercules which head to chop off at each stage of any battle. It is not hard to find a reasonably fast *winning strategy* (i.e. a strategy which ensures that Hercules wins against any hydra). More surprisingly, Hercules cannot help winning:

THEOREM 2. (i) *Every strategy is a winning strategy.*

We can code hydras as numbers and thus talk about battles in the language of first order arithmetic. We cannot formalise Theorem 2(i) as a statement of this language, as strategies are infinitary objects. However, we can if we restrict ourselves to recursive strategies. In that case:

THEOREM 2. (ii) *The statement "every recursive strategy is a winning strategy" is not provable from* P.

In proving the theorems we rely on work of Ketonen and Solovay [3] on ordinals below ε_0, which in turn develops earlier work by the present authors, Harrington, Wainer, and others. Gentzen [1] showed that using transfinite induction on ordinals below ε_0 one can prove the consistency of P, and the Ketonen–Solovay machinery we use here can be viewed as illuminating in more detail the relationship between ε_0 and P.

(*Note:* Goodstein proved the following: if $h : N \to N$ is a non-decreasing function, define an h-Goodstein sequence $b_0, b_1, \ldots$ by letting b_{i+1} be the result of replacing every $h(i)$ in the base $h(i)$ representation of b_i by $h(i+1)$, and subtracting 1. Then the statement "for every non-decreasing h, every h-Goodstein sequence eventually reaches 0" is equivalent to transfinite induction below ε_0.)

To prove Theorem 1 we first define the base n representation more formally, at the same time defining the ordinal $o_n(m)$, in Cantor Normal Form, which results from replacing every n in the base n representation of m by ω.

Suppose $m, n \in N$ (the set of natural numbers), $n > 1$ and

$$m = n^k a_k + n^{k-1} a_{k-1} + \ldots + n a_1 + a_0 .$$

For $x \in N$ or $x = \omega$, set

$$f^{m,n}(x) = \sum_{i=0}^{k} a_i x^{f^{i,n}(x)} .$$

(This definition is by induction on m, starting with $f^{0,n}(x) = 0$.) Then for $m > 0$, $G_n(m) = f^{m,n}(n+1) - 1$ and $o_n(m) = f^{m,n}(\omega)$. Set $G_n(0) = o_n(0) = 0$. Finally for $n \in N$ we define an operation $\langle\alpha\rangle(n)$ on ordinals $\alpha < \varepsilon_0$ by induction on α:

$$\langle 0\rangle(n) = 0 , \quad \langle\beta+1\rangle(n) = \beta$$

and for $\delta > 0$,

$$\langle\omega^\delta(\beta+1)\rangle(n) = \omega^\delta\beta + \omega^{\langle\delta\rangle(n)} n + \langle\omega^{\langle\delta\rangle(n)}\rangle(n) .$$

LEMMA 3. (i) *For* $m \geqslant 0$, $n > 1$, *if* $\alpha = o_{n+1}(m)$ *then* $o_{n+1}(m \dot{-} 1) = \langle\alpha\rangle(n)$.

(ii) *For* $n > 1$, $\langle o_n(m)\rangle(n) = o_{n+1}(G_n(m))$.

Proof. (i) Consider the base $n+1$ representation of m: let

$$m = a_p(n+1)^{f^{p,n+1}(n+1)} + a_{p-1}(n+1)^{f^{p-1,n+1}(n+1)} + \ldots + a_0(n+1)^{f^{0,n+1}(n+1)},$$

with $0 \leqslant a_i \leqslant n$, and (since we may suppose $m \neq 0$) let j be minimal such that $a_j \neq 0$. The result is clear if $j = 0$ so we may assume that $j > 0$ and that the result holds for all $0 < m' < m$. Then

$$o_{n+1}(m \dot{-} 1) = \left(\sum_{i=j+1}^{p} \omega^{f^{i,n+1}(\omega)} a_i\right) + \omega^{f^{j,n+1}(\omega)}(a_j - 1)$$
$$+ o_{n+1}\left(n \cdot (n+1)^{f^{j,n+1}(n+1)-1}\right) + o_{n+1}\left((n+1)^{f^{j,n+1}(n+1)-1} - 1\right),$$

whilst

$$\langle\alpha\rangle(n) = \left(\sum_{i=j+1}^{p} \omega^{f^{i,n+1}(\omega)} a_i\right) + \omega^{f^{j,n+1}(\omega)}(a_j - 1) + \omega^{\langle f^{j,n+1}(\omega)\rangle(n)} n + \langle\omega^{\langle f^{j,n+1}(\omega)\rangle(n)}\rangle(n).$$

Using the inductive hypothesis it is easy to see that these are equal.

(ii) Let $m = \sum_{i=j}^{p} b_i n^{f^{i,n}(n)}$ where $0 \leqslant b_i < n$ and $b_j \neq 0$. If $j = 0$ then it is clear that $\langle o_n(m)\rangle(n) = o_{n+1}(G_n(m))$ so assume $j > 0$. Then

$$\langle o_n(m)\rangle(n) = \left(\sum_{i=j+1}^{p} \omega^{f^{i,n}(\omega)} b_i\right) + \omega^{f^{j,n}(\omega)}(b_j - 1) + \omega^{\langle f^{j,n}(\omega)\rangle(n)} n + \langle\omega^{\langle f^{j,n}(\omega)\rangle(n)}\rangle(n)$$

and

$$o_{n+1}(G_n(m)) = \left(\sum_{i=j+1}^{p} \omega^{f^{i,n}(\omega)} b_i\right) + o_{n+1}\left((n+1)^{f^{j,n}(n+1)} b_j - 1\right)$$
$$= \left(\sum_{i=j+1}^{p} \omega^{f^{i,n}(\omega)} b_i\right) + \omega^{f^{j,n}(\omega)}(b_j - 1) + o_{n+1}\left((n+1)^{f^{j,n}(n+1)-1} n\right)$$
$$+ o_{n+1}\left((n+1)^{f^{j,n}(n+1)-1} - 1\right).$$

By (i)

$$o_{n+1}\left((n+1)^{f^{j,n}(n+1)-1} n\right) = \omega^{\langle f^{j,n}(\omega)\rangle(n)} n$$

and

$$o_{n+1}\left((n+1)^{f^{j,n}(n+1)-1} - 1\right) = \langle\omega^{\langle f^{j,n}(\omega)\rangle(n)}\rangle(n),$$

which gives the required result.

Thus for each Goodstein sequence $b_0, b_1, b_2, \ldots$ there is a corresponding sequence

$$o_n(b_0),\ o_{n+1}(b_1),\ o_{n+2}(b_2),\ \ldots$$

of ordinals. (E.g. in the example given before we would have

$$\omega^{\omega^{\omega+1}}+\omega^{\omega+1}+\omega,\quad \omega^{\omega^{\omega+1}}+\omega^{\omega+1}+2,\quad \omega^{\omega^{\omega+1}}+\omega^{\omega+1}+1,\quad \omega^{\omega^{\omega+1}}+\omega^{\omega+1},\dots.)$$

If we write $\langle\alpha\rangle(n_1, n_2, \dots, n_k)$ for $\langle\dots\langle\langle\alpha\rangle(n_1)\rangle(n_2)\dots\rangle(n_k)$ we can write this sequence as

$$o_n(b_0) = \alpha,\ \langle\alpha\rangle(n),\ \langle\alpha\rangle(n, n+1),\ \langle\alpha\rangle(n, n+1, n+2),\ \dots .$$

It is not hard to see that for any $\alpha < \varepsilon_0$ and $n \in N$,

$$\langle\alpha\rangle(n) < \alpha \quad \text{if } \alpha > 0 .$$

Now we already have Theorem 1 (i) (following Goodstein). For suppose m, n were such that the Goodstein sequence for m starting at n was always positive. Then the corresponding sequence of ordinals would be an infinite, strictly decreasing sequence which is an impossibility (by transfinite induction below ε_0).

In order to prove (ii) we introduce the Ketonen–Solovay machinery (see [3], [4] for details). First we define another operation $\{\alpha\}(n)$ for $\alpha < \varepsilon_0$ and $n \in N$ by induction on α:

$$\{0\}(n) = 0,\ \{\beta+1\}(n) = \beta,\ \{\omega^{\gamma+1}(\beta+1)\}(n) = \omega^{\gamma+1}\beta+\omega^{\gamma}n,$$

and for limit δ,

$$\{\omega^{\delta}(\beta+1)\}(n) = \omega^{\delta}\beta+\omega^{\{\delta\}(n)} .$$

Note that $\{\alpha\}(n) < \alpha$ for $\alpha > 0$. Now define the notion of α-large finite sets for $0 < \alpha < \varepsilon_0$ by induction on α: If $X \subset N$ is finite, enumerate the elements of X in ascending order as $X_0, X_1, \dots, X_{|X|-1}$.

X is 1-large if and only if $|X| \geqslant 2$;

X is α-large if and only if $X-\{X_0\}$ is $\{\alpha\}(X_1)$-large.

Now write $\{\alpha\}(n_1, \dots, n_k)$ for $\{\dots\{\{\alpha\}(n_1)\}(n_2)\dots\}(n_k)$. Then by induction on α we can show that X is α-large if and only if $\{\alpha\}(X_1, X_2, \dots, X_{|X|-1}) = 0$. For details see Lemma 11 of [4].

Define $\omega_0 = \omega$, $\omega_{n+1} = \omega^{\omega_n}$.

The above concepts can be put in the language of first order arithmetic (with ordinals $< \varepsilon_0$ replaced by suitable notations for them) and thus make sense in a nonstandard model of P (see [4]).

THEOREM 4 (Ketonen–Solovay; see [3], [4]). (i) *The function* $Y(a, b) =$ *the greatest* c *such that* $[a, b]$ *is* ω_c*-large is an indicator for models of* P.

(ii) *The statement* $\forall a\, \forall c\, \exists b$ *(*$[a, b]$ *is* ω_c*-large) is independent of* P *and is equivalent in* P *to* $\mathrm{Con}(\mathrm{P}+T_1)$ *where* T_1 *is the set of the true* Π_1 *sentences.*

(iii) *The functions* $g_n(x) =$ *least* $y \geqslant x$ *such that* $[x, y]$ *is* ω_n*-large are provably (in* P*) total recursive functions, and for any provably total recursive function* f *there exists* $n \in N$ *such that* $f(x) < g_n(x)$ *for all sufficiently large* $x \in N$.

Write $\beta \underset{n}{\rightarrow} \alpha$ if and only if for some $j_1, \ldots, j_k \leqslant n$,

$$\alpha = \{\beta\}(j_1, \ldots, j_k);$$

$\beta \underset{n}{\Rightarrow} \alpha$ if and only if the same holds with $j_1 = \ldots = j_k = n$.

The following lemma is a standard application of these concepts.

LEMMA 5. (i) *If $\beta \underset{n}{\Rightarrow} \alpha$ and $n > 0$ then $\omega^\beta \underset{n}{\Rightarrow} \omega^\alpha$.*

(ii) *If $0 < i < j \leqslant n$ then $\{\beta\}(j) \underset{n}{\Rightarrow} \{\beta\}(i)$.*

(iii) *$\beta \underset{n}{\Rightarrow} \alpha$ if and only if $\beta \underset{n}{\rightarrow} \alpha$.*

(iv) *Suppose $\beta = \omega^{\beta_1} + \ldots + \omega^{\beta_n}$, $\gamma = \omega^{\gamma_1} + \ldots + \omega^{\gamma_m}$, and*

$$\beta_1 \geqslant \ldots \geqslant \beta_n \geqslant \gamma_1 \geqslant \ldots \geqslant \gamma_m.$$

Then if $\gamma \underset{n}{\rightarrow} \delta$ then $\beta + \gamma \underset{n}{\rightarrow} \beta + \delta$. In particular $\beta + \gamma \underset{n}{\rightarrow} \beta$.

Having introduced this machinery we shall apply it to link the operations $\{\alpha\}(n)$ and $\langle\alpha\rangle(n)$ and hence obtain our result.

LEMMA 6. *Suppose $\beta \underset{n}{\rightarrow} \alpha$ and $0 < n \leqslant n_1 < n_2 < \ldots < n_k$. Then*

$$\{\beta\}(n_1, \ldots, n_k) \geqslant \{\alpha\}(n_1, \ldots, n_k).$$

Proof. The proof is by induction on β. Assume the result holds below β. By Lemma 5 (iii),

$$\beta \underset{n_1}{\rightarrow} \alpha \underset{n_1}{\rightarrow} \{\alpha\}(n_1)$$

so $\{\beta\}(n_1) \underset{n_1}{\rightarrow} \{\alpha\}(n_1)$. Hence by inductive hypothesis

$$\{\beta\}(n_1, n_2, \ldots, n_k) \geqslant \{\alpha\}(n_1, n_2, \ldots, n_k).$$

PROPOSITION 7. *For all $\alpha < \varepsilon_0$ and $j \in N$, $\langle\alpha\rangle(j) \underset{j}{\rightarrow} \{\alpha\}(j)$.*

Proof by induction on α. If α is 0 or a successor, the result is trivial. If $\alpha = \omega^{\gamma+1}(\beta+1)$ then from the definitions $\{\alpha\}(j) = \omega^{\gamma+1}\beta + \omega^\gamma j$, and $\langle\alpha\rangle(j) = \{\alpha\}(j) + \langle\omega^\gamma\rangle(j)$. Applying Lemma 5 (iv), $\langle\alpha\rangle(j) \underset{j}{\rightarrow} \{\alpha\}(j)$.

If $\alpha = \omega^\delta(\beta+1)$, δ limit, then by inductive hypothesis

$$\langle\delta\rangle(j) \underset{j}{\rightarrow} \{\delta\}(j).$$

By Lemma 5 (i), $\omega^{\langle\delta\rangle(j)} \underset{j}{\to} \omega^{\{\delta\}(j)}$. Thus

$$\langle\alpha\rangle(j) = \omega^{\delta}\beta + \omega^{\langle\delta\rangle(j)} j + \langle\omega^{\langle\delta\rangle(j)}\rangle(j) \underset{j}{\to} \omega^{\delta}\beta + \omega^{\langle\delta\rangle(j)} \qquad \text{by Lemma 5 (iv)}$$
$$\underset{j}{\to} \omega^{\delta}\beta + \omega^{\{\delta\}(j)} = \{\alpha\}(j).$$

PROPOSITION 8. *Let $b_0, b_1, b_2, \ldots$ be the Goodstein sequence for m starting at n and let k be minimal such that $b_k = 0$. Then $[n-1, n+k]$ is $o_n(m)$-large.*

Proof. Consider the corresponding sequence of ordinals

$$o_n(m) = o_n(b_0) = \alpha$$
$$o_{n+1}(b_1) = \langle\alpha\rangle(n)$$
$$o_{n+2}(b_2) = \langle\alpha\rangle(n, n+1)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$
$$o_{n+k}(b_k) = o_{n+k}(0) = 0 = \langle\alpha\rangle(n, n+1, \ldots, n+k).$$

By Lemma 6 and Proposition 7,

$$\{\alpha\}(n, n+1, \ldots, n+k) \leqslant \{\langle\alpha\rangle(n)\}(n+1, \ldots, n+k)$$
$$\leqslant \{\langle\alpha\rangle(n, n+1)\}(n+2, \ldots, n+k)$$
$$\leqslant \ldots \leqslant \langle\alpha\rangle(n, \ldots, n+k) = 0.$$

Hence $[n-1, n+k]$ is α-large.

Now to prove (ii) of Theorem 1 suppose we had

$$\mathrm{P} \vdash \forall m \, \exists k \, m_k = 0. \tag{*}$$

By Theorem 4 and the methods of indicator theory (see [5]) we can find $M \models \mathrm{P}$ and nonstandard $c \in M$ such that

$$M \models \neg \exists y \; ([1, y] \text{ is } \omega_c\text{-large}).$$

(Briefly, this is done by taking a countable nonstandard model J of P and nonstandard $c, a \in J$ such that $Y(c, a)$ is nonstandard but less than $c-1$, where Y is as in Theorem 4 (i). Now the indicator Y having nonstandard value on (c, a) means precisely that there is an initial segment of J which is a model of P and lies "between" c and a, that is, contains c but not a. We can let M be such an initial segment.)

In M, take $d = 2^{2^{\cdot^{\cdot^{2}}}}$ with c iterated exponentiations, so $o_2(d) = \omega_c$. By (*) take $e \in M$ such that $d_e = 0$. Since the proof of Proposition 8 can be carried out in P (see the discussion preceding Lemma 4 in [4]), we have in M

$$[1, 2+e] \text{ is } \omega_c\text{-large},$$

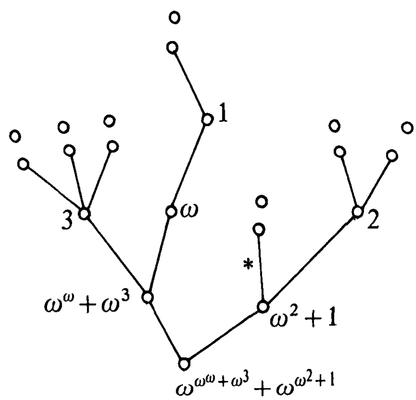
a contradiction.

We sketch the proof of Theorem 2, which is similar to that of Theorem 1. First assign to each node in a given hydra an ordinal below ε_0 as follows:

To each top node assign 0.

To each other node assign $\omega^{\alpha_1}+\ldots+\omega^{\alpha_n}$, where $\alpha_1 \geqslant \ldots \geqslant \alpha_n$ are the ordinals assigned to the nodes immediately "above". ($\omega^0 = 1$).

Thus our original example would have the assignments



The ordinal of a hydra is the ordinal assigned to its root. For any strategy σ, we can define an operation $[\alpha]_\sigma(n)$ which maps the ordinal of the hydra after stage $n-1$ to the ordinal of the hydra after stage n, where σ is the strategy being used.

To show Theorem 2 (i) it will suffice for the reader to check that for any strategy σ, any $0 < \alpha < \varepsilon_0$ and $n \in N$,

$$[\alpha]_\sigma(n) < \alpha .$$

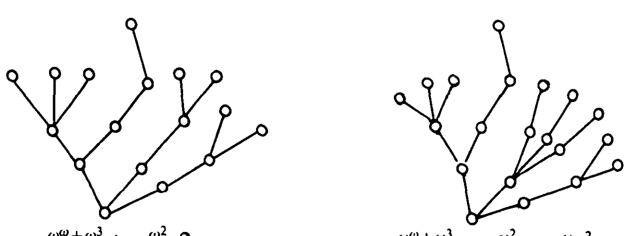
For (ii) we shall produce a recursive strategy τ for which

$$[\alpha]_\tau(n) = \{\alpha\}(n+1) .$$

Then a proof like that of Theorem 1 will show that in P we cannot prove that τ is a winning strategy.

An algorithm for τ is as follows: starting from the root, we travel "up" the tree in such a way that, having reached a node, we travel to the node immediately above it which has minimal assigned ordinal among all the nodes immediately above it. (If more than one of them has minimal ordinal we choose, say, the leftmost.) Eventually we reach a top node and the head it is attached to is the one to chop off.

Thus in the previous diagram the head determined by τ is starred, and the battle determined by τ begins thus:



after stage 1 after stage 2

(*Note:* A proof that τ is a winning strategy is equivalent to a proof of "ε_0-induction with respect to the predecessor function $p(\alpha, n) = \{\alpha\}(n)$" which amounts in turn to a proof that the function $\lambda nx . g_n(x)$ of Theorem 4 (iii) is total recursive, and this of course is impossible in P.)

REMARK. Let $I\Sigma_k$ denote Peano's axioms with induction restricted to Σ_k' formulae. Then using results in [4] we can refine Theorem 1 to give, for $k \in N, k \geqslant 1$,

THEOREM 1'. (i) *For each fixed* $p \in N$, $I\Sigma_k \vdash \forall m, n > 1$ (*if* $m < n^{n^{\cdot^{\cdot^{\cdot^{np}}}}}$ (*where n occurs k times*) *then the Goodstein sequence for m starting at n eventually hits zero*).

(ii) $I\Sigma_k \nvdash \forall m, n > 1$ (*if* $m < n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}$ (*where n occurs* $k+1$ *times*) *then the Goodstein sequence for m starting at n eventually hits zero*).

Similarly if we restrict ourselves in Theorem 2 to hydras of height $k+1$ (i.e. no node is more than $k+1$ segments away from the root) then we cannot prove that "every recursive strategy is a winning strategy" using just $I\Sigma_k$. In particular the function giving the lengths of battles for hydras of height 2 with strategy τ is not provably total in $I\Sigma_1$ and hence is not primitive recursive.

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Department of Mathematics,
The University of Manchester,
Manchester M13 9PL.