SEQUENCES — Fibonacci

Definition

In the Fibonacci sequence, each term is the sum of the previous two.

Recurrence relation

The sequence (F_n) of Fibonacci numbers is defined by:

$F_1 = F_2 = 1$

$F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$

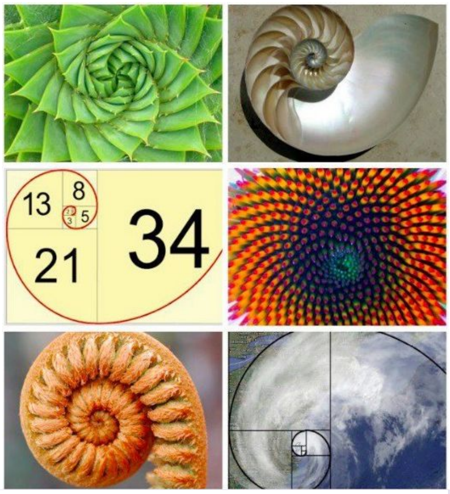
Formula

$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$ for $n \geq 1$

History

The Fibonacci sequence is named after Leonardo of Pisa, an Italian mathematician whose nickname was Fibonacci, and who lived in the early 13th century. But this sequence was known in India hundreds of years before.

The Fibonacci sequence is very interesting to study, since it has connections to many branches of mathematics.

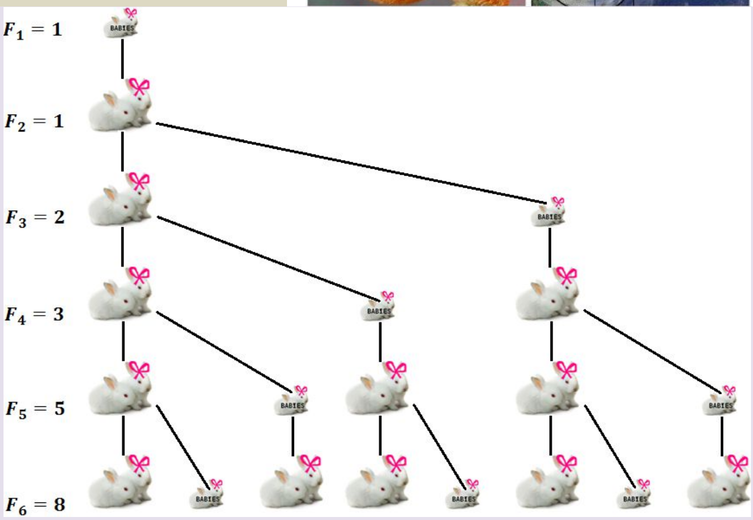


The rabbit problem

In his book, Fibonacci studied how fast rabbits breed in ideal circumstances. He imagined a newly born pair of rabbits, one male and one female were placed in a field and made the following assumptions about the mating habits of rabbits:

- Rabbits mate at exactly the age of one month and mate every month after that.
- Each adult pair produces a mixed pair of baby rabbits.
- The gestation period for rabbits is exactly one month.
- Rabbits never die.

Fibonacci asked how many rabbits a single pair can produce after a year.



Starting with one pair, the sequence we generate is the Fibonacci sequence.

The golden ratio $\frac{F_{n+1}}{F_n} \to \varphi$

We consider the Fibonacci sequence defined by:

$F_1 = F_2 = 1$

$F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$

Let (V_n) be the sequence defined by: $V_n = \frac{F_{n+1}}{F_n}$ for $n \geq 1$.

(The numbers V_n are the ratios of successive Fibonacci numbers.)

1) Complete the following table:

n	1	2	3	...	11
F_n	1	1			
V_n					

2) Show that (V_n) satisfies the recurrence relation $V_{n+1} = 1 + \frac{1}{V_n}$.

3) Verify that $\varphi = \frac{1+\sqrt{5}}{2}$ is a solution of the equation $\varphi^2 - \varphi = 1$.

4) Prove that $V_{n+1} - \varphi = \frac{(\varphi-1)(\varphi-V_n)}{V_n}$ for any integer n.

Deduce that $|V_{n+1} - \varphi| \leq 0.7|V_n - \varphi|$.

5) Deduce that $|V_n - \varphi| \leq 0.7^n$.

What number does V_n approach as n increases ?

φ is called the golden ratio.

Algorithm

input:

the number n of the term searched, $n \geq 3$

output:

the nth term of the Fibonacci sequence

```
fibo(n)
Prgm
1 → a
1 → b
2 → k
While k < n
   a + b → c
   b → a
   c → b
   k + 1 → k
EndWhile
Disp c
EndPrgm
```